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LETTER TO THE EDITOR

A crossover from Fermi-liquid to non-Fermi-liquid behaviour in a solvable one-dimensional model

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Abstract. We consider a quantum many-body problem in one dimension described by a Jastrow-type wavefunction, characterized by an exponent λ and a parameter γ. In the limit $\gamma = 0$ the model becomes identical to the well known $1/r^2$ pair-potential model; γ is shown to be related to the strength of a many-body correction to the $1/r^2$ interaction. Exact results for the one-particle density matrix are obtained for all γ when $\lambda = 1$, for which the $1/r^2$ part of the interaction vanishes. We show that with increasing γ, the Fermi-liquid state (at $\gamma = 0$) crosses over to distinct γ-dependent non-Fermi-liquid states, characterized by effective 'temperatures'.

A special class of Jastrow-type wavefunctions [1]

$$\Psi(u_1, \ldots, u_N) = C \prod_{1 \leqslant a < b \leqslant N} |u_a - u_b|^\lambda \prod_{c=1}^{N} \mathrm{e}^{-V(u_c)} \tag{1}$$

appear frequently in quantum many-body problems. In one dimension, many-body Hamiltonians with pair potentials of the form $1/r^2$ (or its periodic equivalent $1/\sin^2 r$), where $r = (u_a - u_b)$, have exact ground-state wavefunctions of the Jastrow form [2–5]. It has been proposed that variational wavefunctions of this type give reasonably good descriptions of models for strongly interacting fermions [6–11]. It is therefore of interest to investigate the properties of such wavefunctions as exactly as possible. Of particular interest is the question of whether such a wavefunction describes how interaction can change a Fermi liquid into a non-Fermi-liquid state.

In the solution of the $1/r^2$ model, the parameter λ in (1) is related to the strength of the $1/r^2$ pair potential; in particular, $\lambda = 1$ corresponds to the free-fermion case. Unfortunately, the exact density matrix for this model can be obtained only for a few special values of λ [2, 12], so the question of the nature of the crossover from the free-fermion to the interacting non-Fermi-liquid state cannot be addressed exactly in this model. In the present work, we consider a one-dimensional wavefunction of the above form which can be considered as a generalization of the wavefunction corresponding to the $1/r^2$ model. In addition to the parameter λ, our model contains an additional parameter γ which we show to be related to the strength of a many-body correction to the $1/r^2$ interaction. We obtain the one-particle density matrix for this model exactly for all γ, for the particular case of $\lambda = 1$. Thus $\gamma = 0$ represents the free-fermion case. We show that with increasing γ, the Fermi liquid state is destroyed by an effective non-zero 'temperature' induced by interaction.

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It is quite remarkable that the square of the expression (1) with $V(u) = \lambda u^2$ and $\lambda = \frac{1}{2}$, 1 or 2 is identical in form to the joint probability distribution of the eigenvalues of random matrices in a Gaussian ensemble [13]. The two-level correlation function for such an ensemble is known exactly from the theory of random matrices, and the analogy was exploited by Sutherland [2] to obtain the one-particle density matrix for the $1/r^2$ model. (In general, for an arbitrary $V(u)$, the n-point correlation function can be obtained explicitly in terms of orthogonal polynomials defined with $V(u)$ as its weight factor.) In the present work we exploit a similar analogy to a recently introduced family of random matrices [14] which may be considered as a generalization of the conventional Gaussian ensemble. We consider a wavefunction given by (1) with a more general $V(u)$ given by

$$V(u) = \frac{1}{2\gamma}\left[\sinh^{-1}[(\gamma\omega)^{1/2}u]\right]^2 + \frac{1}{2}\ln\left[\vartheta_4\left(\frac{\pi}{\gamma}\sinh^{-1}[(\gamma\omega)^{1/2}u];\, p\right)\right] \quad (2)$$

characterized by a single parameter γ. Here $\vartheta_4(x;p)$ is the Jacobi theta function [15], $p = e^{-\pi^2/\gamma}$ and ω has the dimension of $1/[\text{length}]^2$. For $\gamma = 0$, $V(u) = \frac{1}{2}\omega u^2$, and the wavefunction reduces to the well known solution of the $1/r^2$ pair potential. The case $\lambda = 1$ then represents a free-fermion problem (with the choice of Fermi statistics), with the one-particle density matrix given by $\sin[\pi D r]/(\pi r)$, D being the density of particles. The parameter γ 'deforms' the harmonic well into a weakly confining $[\ln u]^2$ term for large enough u, as shown in figure 1. We will show that this deformation leads to a qualitative change in the density matrix; for $\lambda = 1$ this corresponds to a change from a Fermi-liquid to a non-Fermi-liquid state.

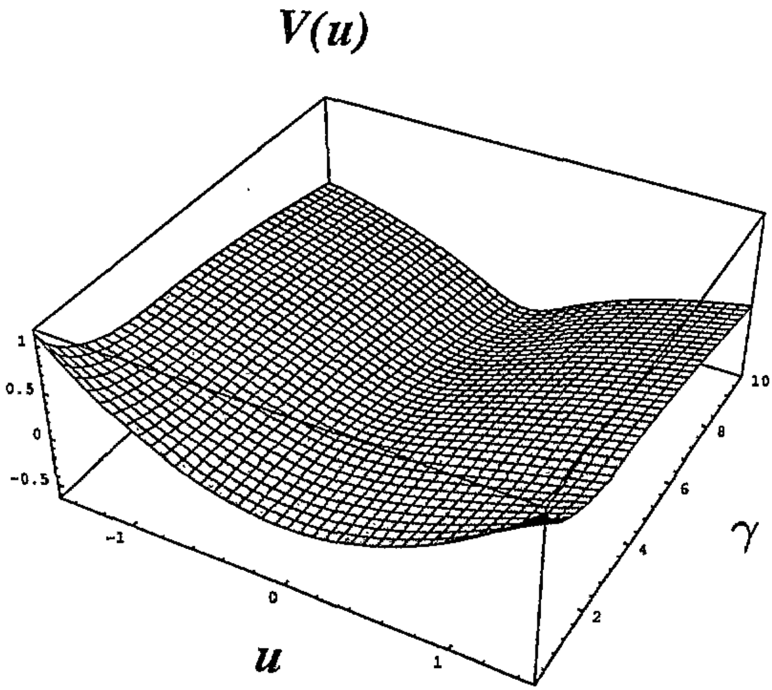

Figure 1. $V(u)$ characterizing the model wavefunction considered, as given by equation (2), for various values of γ. The parameter ω has been set equal to unity.

In order to understand the role of the parameter γ, let us concentrate for the moment on the small-γ ($\ll \pi^2$) limit where the second term in equation (2) can be neglected, and the Hamiltonian corresponding to the above wavefunction has a simple form. In this limit, to leading order in γ, the Schrödinger equation (in units where $\hbar^2/2m = 1$) can be written as

$$\begin{aligned}\frac{1}{\Psi}\sum_k \frac{\partial^2}{\partial u_k^2}\Psi &= \lambda(\lambda-1)\sum_{j\neq k}\frac{1}{(u_k-u_j)^2}\\ &+\omega^2[1+2\gamma]\sum_k (u_k)^2 - \omega[N+\lambda N(N-1)]\\ &+\frac{2}{3}\lambda\gamma\omega^2\sum_{j\neq k}\frac{1}{u_k-u_j}[(u_k)^3-(u_j)^3] - \frac{4}{3}\gamma\omega^3\sum_k (u_k)^4. \end{aligned} \tag{3}$$

Note that for $\gamma = 0$, the Hamiltonian reduces to the $1/(u_k - u_j)^2$ pair potential, with energy $E = \omega[N + \lambda N(N-1)]$ [2,3]. For $\lambda = 1$, with the choice of Fermi statistics, this becomes a free-fermion problem. However, for $\gamma \neq 0$, a many-body correction term survives even for $\lambda = 1$. Nevertheless, the density matrix for $\lambda = 1$ can still be found exactly for all γ. We will show that it corresponds to a non-Fermi-liquid state, γ playing the role of an effective temperature.

As mentioned before, we obtain the exact density matrix corresponding to the wavefunction defined by equations (1) and (2) for $N \to \infty$ and for all γ, by exploiting the analogy of the present problem with the random matrix model recently constructed in [14]. It is given by

$$\rho(u,v) = f(\gamma)Q(\mu,\nu)\frac{\vartheta_1((\pi(\mu-\nu)/2\gamma);p)}{u-v} \tag{4}$$

where

$$Q(\mu,\nu) = \frac{\vartheta_4((\pi(\mu+\nu)/2\gamma);p)}{\vartheta_4^{1/2}(\pi\mu/\gamma;p)\,\vartheta_4^{1/2}((\pi\nu/\gamma);p)} \tag{5}$$

and

$$\mu = \sinh^{-1}(\sqrt{\gamma\omega}u) \qquad \text{and} \qquad \nu = \sinh^{-1}(\sqrt{\gamma\omega}v)$$

$\vartheta_1(x;p)$ is a Jacobi theta function [15] and $f(\gamma)$ is a known function of γ. For $\gamma \ll \pi^2$, a simpler form for the density matrix is obtained.

$$\rho(u,v) \simeq \frac{1}{\pi}\frac{\sin\left[(\pi/2\gamma)\left(\sinh^{-1}\left[(\omega\gamma)^{1/2}u\right] - \sinh^{-1}\left[(\omega\gamma)^{1/2}v\right]\right)\right]}{u-v}. \tag{6}$$

In the limit $\gamma \to 0$, in terms of the density at the origin $D_0 = \frac{1}{2}(\omega/\gamma)^{1/2}$, this reduces to the free-fermion density matrix $\rho(u-v) = \sin[D_0\pi(u-v)]/\pi(u-v)$. These oscillations are the characteristic signature of a normal Fermi system; its Fourier transform—the momentum distribution—is the familiar step function. On the other hand for increasing γ, these oscillations begin to die out, as shown for the normalized $\rho(u,0)$ in figure 2, destroying the Fermi-liquid-behaviour. (We note that the density matrix is not translationally invariant, and the non-Fermi-liquid state is different from other translationally invariant non-Fermi liquid states like the Luttinger liquid.) The Fourier transform of $\rho(u,v)$ would involve two external momenta and this makes comparison with the Fermi liquid momentum distribution difficult.

We shall instead consider the Fourier transform of $\rho(u,0)$, $n_k = \int_{-\infty}^{+\infty} \mathrm{d}u\ \mathrm{e}^{iku}\rho(u,0)$. With the change of variable $y = \sinh^{-1}(2\gamma D_0 u)$, we have

$$n_k = \frac{1}{\pi}\int_0^\infty \mathrm{d}y \coth y\left[\sin\frac{\pi}{2\gamma}\left(\frac{|k|}{\pi D_0}\sinh y + y\right) - \sin\frac{\pi}{2\gamma}\left(\frac{|k|}{\pi D_0}\sinh y - y\right)\right]. \tag{7}$$

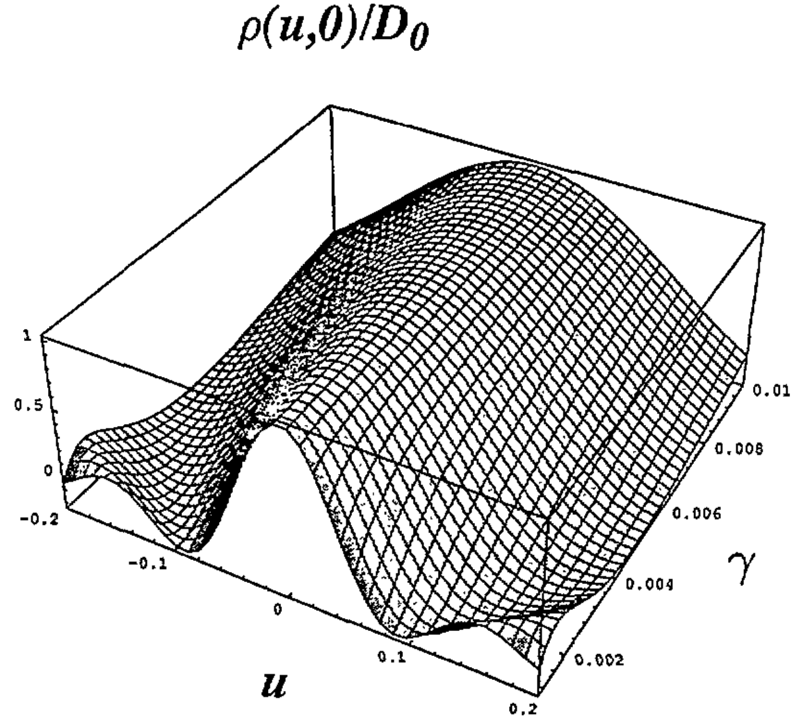

Figure 2. The one-particle normalized density matrix $\rho(u,0)/D_0$ as obtained from equation (6), for values of γ between 0.001 and 0.01. The parameter ω has been set equal to unity.

For small enough γ we can replace $\sinh y$ by y within the sine function, and $\coth y$ by $1/\sinh y$. The resulting distribution is given by

$$n_k = n\left[\frac{\pi}{\gamma}\left(\frac{|k|}{D_0} - \pi\right)\right] - n\left[\frac{\pi}{\gamma}\left(\frac{|k|}{D_0} + \pi\right)\right] \tag{8}$$

where $n[x] = 1/[e^x + 1]$ is the Fermi function. As expected, this reduces to the step of the free fermions when $\gamma = 0$. Moreover, we find from the explicit expression (8) that γ plays the role of an effective 'temperature'. This may be seen in a more physical way by considering the static form factor $S_k = 1 - b_k$, where $b_k = \int_{-\infty}^{\infty} du\ e^{iku}[\rho(u,0)]^2$, is the Fourier transform of the pair correlation function which is the probability of finding a particle at u given that there is a particle located at zero. According to a standard sum-rule argument, the dispersion relation of the elementary excitation in the long wave-length limit can be expressed as $\epsilon_k = k^2/2mS_k$ [16]. For $\gamma = 0$ we have a sound-like dispersion since $S_k \sim k$, $(k \to 0)$, while for $\gamma \neq 0$ it can be shown that $S_0 =$ constant [17] which is consistent with the interpretation of γ as an effective temperature. In this limit $\epsilon_k/k \to 0\ \ k \to 0$.

In summary, we have considered a one-dimensional quantum many-body problem described by a Jastrow-type wavefunction characterized by an exponent λ and a parameter γ. We show that our model is a generalization of the $1/r^2$ pair-potential model considered by Calogero and Sutherland, which is obtained in the limit $\gamma = 0$. We obtain the exact one-particle density matrix for all γ for the case $\lambda = 1$ where the $1/r^2$ interaction vanishes. For $\gamma = 0$, and the choice of Fermi statistics, this becomes a free-fermion problem, and we recover the step function for the momentum distribution. For $\gamma \neq 0$, an interaction term survives for the case $\lambda = 1$ and the resulting momentum distribution is smeared out, destroying the Fermi liquid. The explicit expression for the momentum distribution as a function of γ for small γ shows that the destruction of the Fermi-liquid state occurs as increasing interaction induces an increase in the effective 'temperature' in this regime.

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